MATH 147 Review: Line Integrals

Facts to Know

If f is a real-valued function defined on a smooth curve $C \subseteq \mathbb{R}^2$, then

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\ x'(t)\,dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t), y(t))\ y'(t)\,dt$$

Here $x(t), y(t)$, $t \in [a,b]$ is a parametrization of C (with a given orientation)

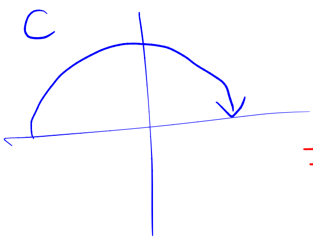
Examples

$\overset{I}{\shortparallel}$

1. Evaluate $\int_C (2 + x^2 y)\, dy$, where C is the upper half of the unit circle $x^2 + y^2 = 1$ with a clockwise orientation.

$$\begin{cases} x(t) = 1\cos(-t+\pi) + 0 \\ y(t) = 1\sin(-t+\pi) + 0 \end{cases} \qquad t \in [0, \pi]$$

C



$$I = \int_0^\pi \left(2 + \cos^2(-t+\pi)\sin(-t+\pi)\right)\cos(-t+\pi)(-1)\,dt$$

$$= \int_0^\pi 2(-1)\cos(-t+\pi) + (-1)\cos^3(-t+\pi)\sin(-t+\pi)\ dt$$

$$= \cancel{\int_0^\pi -2\cos(-t+\pi)}^{\,0} + \int_0^\pi (-1)\cos^3(-t+\pi)\sin(-t+\pi)\,dt \quad \Bigg|\ \begin{array}{l} u = \cos(-t+\pi) \\ du = +\sin(-t+\pi)\,dt \end{array}$$

$$= \cancel{\int_{-1}^{+1} (-1)u^3\,du}^{\,0} = 0$$

2. Evaluate $\int_C 2x\,dx$, where C consists of the arc C_1 of the line segment from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

$$\int_C 2x\,dx = \int_{C_1} 2x\,dx + \int_{C_2} 2x\,dx$$

$$= \int_0^1 2t \cdot 1 \cdot dt + \cancel{\int_0^1 2(1)(0)\,dt}$$

$$= \int_0^1 2t\,dt$$

$$= t^2 \Big|_0^1 = 1^2 - 0^2 = 1$$

C_1

$$\begin{cases} x(t) = (1-t)\,\underline{0} + t\,\underline{1} \\ y(t) = (1-t)\,\underline{0} + t\,\underline{1} \end{cases}$$

$t \in [0,1]$

$$\begin{cases} x(t) = t \\ y(t) = t \end{cases}$$

C_2

$$\begin{cases} x(t) = 1 \\ y(t) = 1+t \end{cases}$$

$t \in [0,1]$